then is

$$\frac{\partial W}{\partial \eta} - W\frac{\partial W}{\partial s} = \epsilon\frac{\eta - 1 + \eta_0}{2}\frac{\partial^2 W}{\partial s^2} \tag{23}$$

$$\epsilon = \frac{2b\sqrt{M^2-1}}{\rho_0 v_0 (\gamma+1)\ell_0 \eta_0} \tag{24}$$

The N wave solution of Eq. (23) is obtained by setting $\epsilon = 0$:

$$W(s,\eta) = -(s/\eta),\ s<\eta^{1/2};\qquad W(s,1) = -s,\ s<1$$

$$W(s,\eta) = 0,\qquad s>\eta^{1/2};\qquad W(s,1)=0,\quad s>1 \tag{25}$$

In Eqs. (25) we have assumed $s>0$ because of the antisymmetry of $W(s,\eta)$ about $s=0$. The detailed behavior of $W(s,\eta)$ in the neighborhood of $s=\eta^{1/2}$ is from Ref. 7,

$$W(s,\eta) = -\frac{1}{2}\eta^{-1/2} \times \left\{1 - \tanh\frac{[(s-\eta^{1/2})/\epsilon] - \frac{1}{2}\eta^{1/2}(\eta - 1 + (\eta_0 - 1)\ln\eta)}{4\eta^{1/2}[(\eta+\eta_0-1)/2]}\right\} \tag{26}$$

The problem of finding a solution of Eq. (23) for large η values that matches solutions (25) and (26) is solved in Ref. 5, i.e.,

$$W = -C\frac{s}{\epsilon\eta^2}\Phi\left(1,\frac{3}{2};-\frac{s^2}{\epsilon\eta^2}\right) \tag{27}$$

where

$$C = 1 - \tanh\tfrac{1}{4} \tag{28}$$

and Φ is the confluent hypergeometric function. With the physical variables we obtain the following expression for v, which also gives us the pressure p through Eq. (18):

$$v = -Cr^{-3/2}s\frac{\ell_0}{c}c^2R_0^{1/2}\frac{\rho_0\ell_0 v_0}{2b}\frac{(M^2-1)^{1/2}}{M^3} \times\Phi\left(1,\frac{3}{2};-\left(\frac{s\ell_0}{c}\right)^2\frac{c^3\rho_0}{2br}\frac{(M^2-1)^{1/2}}{M^3}\right) \tag{29}$$

(In Ref. 5, formula (50), ϵ is missing in the denominator of the argument of Φ.) According to Eq. (29) the s maximum of v decays as r^{-1}.The order of magnitude of r in Eq. (29) is

$$\frac{c_0^2\ell_0^2}{v_0^2R_0 r} = o(\epsilon^2) \tag{30}$$

The quantities v_0 and R_0 in Eq. (29) need some comment. As was mentioned before, for $r=R_0$ the N wave is fully developed. Since for a fully developed N wave the shock amplitude v_0 decreases with $R_0^{-1/4}$, the old-age value of v in Eq. (29), the right-hand side of which contains the product $v_0R_0^{1/2}$, thus decreases with $R_0^{-1/4}$. Thus it is desirable to determine the value of R_0 as exactly as possible. This can be done by the method in Ref. 2 in the following way. The value of v_0, according to Ref. 2, formula (11-10.18) and Eq. (18) herein, is

$$v_0 = \left\{\left[2^{1/4}\left(\mathrm{Max}\int_{-\infty}^{\xi}F_W(x)\mathrm{d}x\right)^{1/2}c(M^2-1)^{-1/8}\right] \div [R_0^{1/2}(R_0-r_0)^{1/4}(\gamma+1)^{1/2}]\right\}\frac{\sqrt{M^2-1}}{M} \tag{31}$$

where $F_W(x)$ is the Whitham F function, which is known if the form of the projectile is known. The limit ξ in this integral depends on R_0 and the N wave can be considered as developed when R_0 has reached a value for which the integral over F_W in Eq. (31) does not grow essentially greater. An uncertainty of 50% in the determination of R_0 gives an uncertainty of about 10% in v in the result [Eq. (29)].

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A Noniterative Finite Difference Method for the Compressible Unsteady Laminar Boundary Layer

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Introduction

In order to determine the friction drag and the rate of heat transfer at the surface of a body in dynamic motion, the unsteady viscous flow must be investigated in detail. Although major progress has been made over the decades in the study of unsteady viscous flows, our information is still not sufficient, due to the many flow-complicating factors. Even for the simpler attached boundary layers, the problem can still be complicated through interaction among the increased dimensionality, nonlinearity, and compressibility.

A variety of successful methods have appeared in the literature for general, unsteady boundary-layer research. A comprehensive list is found in Tellionis.[1] In numerical simulation of the unsteady boundary layers, iterative methods have been widely used due to the nonlinear convection terms in the governing equations.[2,6]

In this Note, the nonlinearity is eliminated by using a known technique of linearizing the general nonlinear implicit finite difference equations without sacrificing accuracy.[3,4] Transient as well as oscillating two-dimensional compressible boundary layers are solved numerically by using a non-

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iterative implicit finite difference scheme that is second-order accurate in both time and space. It is assumed that the compressibility is caused within the boundary layer due to the arbitrary wall temperature, while the external flow is kept incompressible. This method can, of course, be easily applied to incompressible boundary layers.[5]

Numerical Formulation

The unsteady compressible laminar boundary layers are governed by the equations given in Vimala and Nath[6] and Hung.[7] As in these references, we further assume that the gas is perfect and the viscosity is proportional to the absolute temperature. In order to retain the form of basic equations similar to that of incompressible flow, we use the Dorodnitsyn's transformation. After some manipulation, we obtain the governing equations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\left(\frac{\partial U_e}{\partial t}+U_e\frac{\partial U_e}{\partial x}\right)\frac{h}{h_e}+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \tag{2}$$

$$\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+v\frac{\partial h}{\partial y}=\frac{\partial}{\partial y}\left(\frac{1}{Pr}\frac{\partial h}{\partial y}\right) \tag{3}$$

where h and h_e are the enthalpy in the field and at the boundary layer edge, respectively.

We introduce an unsteady coordinate transformation using the dimensionless stream function $f(x,\eta,t)$ and dimensionless enthalpy $g(x,\eta,t)$,

$$x=x,\quad t=t,\quad \eta=y\sqrt{U_e/x},\quad \psi=\sqrt{xU_e}f(x,\eta,t)$$

$$g=\frac{h}{h_e},\quad u=\frac{\partial\psi}{\partial y},\quad v=-\frac{\partial\psi}{\partial x} \tag{4}$$

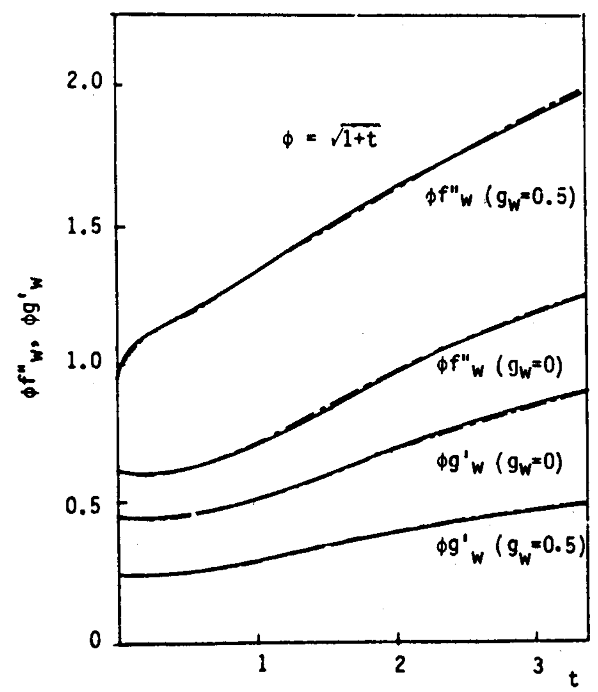

Fig. 1 Skin-friction and wall heat-transfer parameters (—— present investigation, – · – Ref. 6.)

The three differential equations (1-3) can be reduced to two expressions with increased orders, as given by

$$f'''+m_1ff''+m_2f'^2+m_3g+m_4f'+m_5\eta f''$$
$$=m_6\frac{\partial f'}{\partial t}+m_7\left(f'\frac{\partial f'}{\partial x}-f''\frac{\partial f}{\partial x}\right) \tag{5}$$

$$\frac{1}{Pr}g''+m_1fg'+m_5\eta g'=m_6\frac{\partial g}{\partial t}+m_7\left(f'\frac{\partial g}{\partial x}-g'\frac{\partial f}{\partial x}\right) \tag{6}$$

with the renewed boundary conditions

$$\eta=0:\quad f=f'=0,\quad g=g_w$$
$$\eta\to\infty:\quad f'\to 1,\quad g\to 1 \tag{7}$$

In the above equations, the primes denote differentiation with respect to η and the coefficients $m_1,m_2,\ldots,m_7$ are known functions of x and t only. Especially, it is noted that m_7 is equal to x so that at the initial plane $x=0$ it vanishes, eliminating the x derivatives from the equations.

Since the partial differential equations are biparabolic, namely having two parabolic directions x and t, we need initial data in both the time and streamwise direction. As a temporal initial condition for an oscillatory flow, the steady-state solutions can be taken. For transition or starting flow problems, the exact initial flow status is usually known and, hence, no difficulty exists. In obtaining the spatial initial value to be able to march in x direction, a great advantage occurs with the present form of the transformed equations. At the initial plane $x=0$, Eqs. (5) and (6) are reduced to the much simpler monoparabolic form having one marching direction in t only.

First of all, Eqs. (5) and (6) are reduced to first order by introducing three new variables, $f'=u$, $u'=v$, and $g'=h$ (where u, v, and h are redefined). The resulting five first-order equations are solved by advancing the solution in the time coordinate while marching in the x direction simultaneously. Discretization in the time coordinate direction is made by using an incremental value Δ as in $W^{n+1}=W^n+\Delta W^{n+1}$, where W represents such variables as $f,u,v,g,h,$ and t. We make discretization of the time derivative using a generalized time-differencing formula first introduced by Beam and Warming.[3] Specifically, we later choose the three-point backward implicit method of second-order accuracy. The nonlinear terms in the corresponding equations (5) and (6) are linearized, without losing the formal order of accuracy in time, by a similar procedure as in Orlandi and Ferziger.[4] Next, spatial discretization is introduced by using variable step sizes Δx and $\Delta\eta$. Central differencing is made for the spatial derivatives about the point $(i,j-½)$ for the equations $f'=u$, $u'=v$, and $g'=h$ and about the point $(i-½,j-½)$ for the rest of equations.

A set of five implicit finite difference equations is obtained and is written in a 5×5 block tridiagonal matrix form. This sytem of equations is solved without difficulty by using the well-known block elimination method.[9]

Results

To demonstrate the capability of the present numerical method, an unsteady stagnation point flow, which is suddenly accelerated at $t=0$ from a constant speed of the freestream, is considered here. The outer-edge potential flow is given by

$$U_e(x,t)=x(1+t) \tag{8}$$

In Fig. 1, the wall shear stress parameter f'' and wall heat flux parameter g' obtained by the present method offer good comparison with the data available from Vimala and Nath,[6] where two nondimensional wall temperatures $g_w=0$ and 0.5 are used.

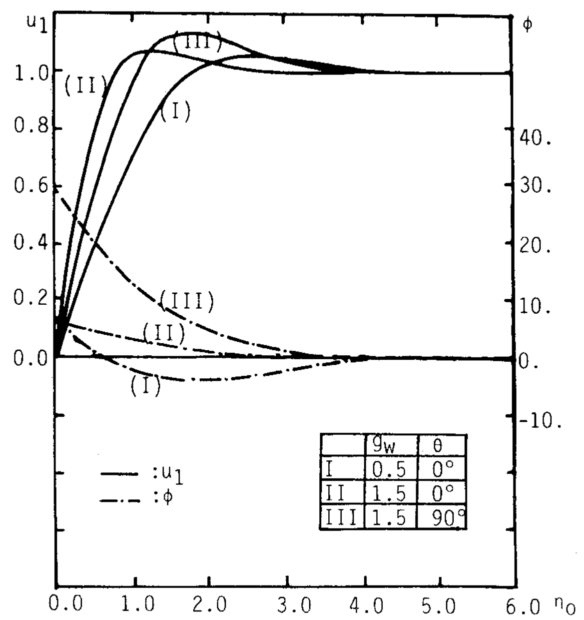

Fig. 2 Amplitude and phase angle of primary velocity fluctuation.

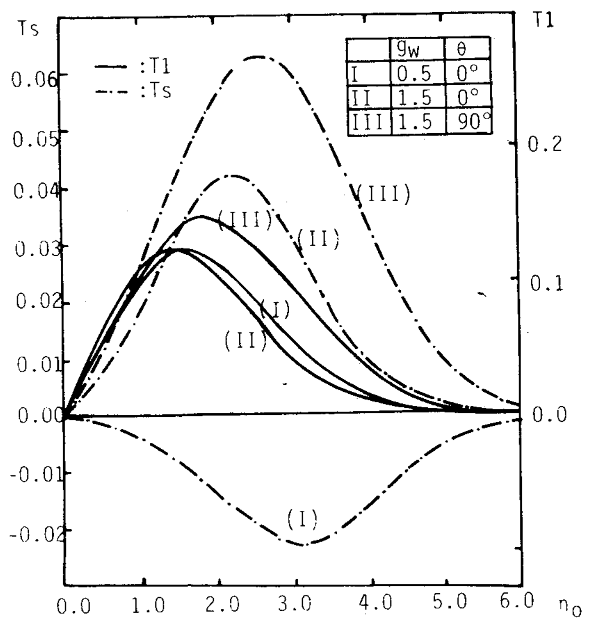

Fig. 3 Amplitude of the primary temperature and the steady streaming temperature component.

For a circular cylinder placed cross stream and oscillating harmonically in the freestream direction with a positive mean velocity, the wall potential flow is given by

$$U_e = 2\sin x(1+\epsilon\sin\omega t) \tag{9}$$

The cylinder wall is heated to a constant temperature $g_w = 0.5$ or 1.5. A characteristic representation of the boundary-layer response can be made by using a Fourier decomposition[8]

$$u = 2\sin x\{u_0 + \epsilon u_1 \sin(\omega t + \phi) + \epsilon^2[u_s + u_2\sin(2\omega t + \phi')] + \ldots\} \tag{10}$$

Figure 2 is a plot of the amplitude u_1 and the phase angle ϕ of the primary velocity fluctuation term. The former displays a characteristic overshoot in the entire boundary layer, whose peak value becomes larger and occurs at a higher η position with increasing azimuthal angle θ. Another characteristic feature is that the flow near the wall leads the outer flow by a phase angle sharply increasing with the azimuthal angle θ of the cylinder. At the higher wall temperature, the overshoots in u_1 and the phase angle are greater than those at the lower wall temperature.

A similar decomposition for the unsteady temperature of the boundary layer leads to Fig. 3. The amplitude of the primary fluctuating temperature shown is intriguing in that the peak values occur quite far off the wall as a result of unsteady convection. The steady streaming temperature T_s indicates that the wall temperature has a large effect on the magnitude and even on the sign of T_s, implying that the heat flux is also very much influenced.

Conclusions

The noniterative numerical formulation presented in this note has been developed to solve the unsteady compressible laminar boundary layer equations efficiently. The exact spatial initial condition provided within the method itself at the initial plane, together with the second-order formal truncation error in time and space, produced a very accurate unsteady solution in comparision with the fully iterative numerical methods. The application made to the oscillating heated cylinder revealed nonsimilar, highly nonlinear unsteady velocity and thermal response within the boundary layer.

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The Second Mode of the Görtler Instability of Boundary Layers

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Introduction

THE Görtler instability of boundary layers produces a secondary flow in the form of counter-rotating vortices

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